OCCUPATION TIMES OF SETS OF INFINITE MEASURE FOR ERGODIC TRANSFORMATIONS

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ABSTRACT. Assume that T is a conservative ergodic measure preserving transformation of the infinite measure space $(X, \mathcal{A}, \mu)$.We study the asymptotic behaviour of occupation times of certain subsets of infinite measure. Specifically, we prove a Darling-Kac type distributional limit theorem for occupation times of barely infinite components which are separated from the rest of the space by a set of finite measure with c.f.-mixing return process. In the same setup we show that the ratios of occupation times of two components separated in this way diverge almost everywhere. These abstract results are illustrated by applications to interval maps with indifferent fixed points.

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1. INTRODUCTION

Let T be a conservative ergodic measure preserving transformation (c.e.m.p.t.) of the σ-finite measure space $(X, \mathcal{A}, \mu)$ with $\mu(X) = \infty$. We are interested in the long term statistical behaviour of occupation times $\mathbf{S}_n(A) := \sum_{k=0}^{n-1} 1_A \circ T^k$, $n \geq 1$, of suitable sets A with $\mu(A) = \infty$. The results we are going to prove in the subsequent sections apply in particular to infinite measure preserving interval maps with indifferent fixed points, and we now illustrate them in this setup. For simplicity we restrict our attention to the prototypical situation of transformations with two full branches (for a more general framework see e.g. [Z1]). As in [T5] we shall consider maps $T : [0, 1] \to [0, 1]$ which fulfil the following conditions for some $c \in (0, 1)$:

(1) The restrictions $T\mid_{(0,c)}$, $T\mid_{(c,1)}$ are $\mathcal{C}^2$-diffeomorphisms onto $(0, 1)$, admitting $\mathcal{C}^2$-extensions to the respective closed intervals;
(2) $T' > 1$ on $(0, c] \cup [c, 1)$ and $T'(0) = T'(1) = 1$;
(3) T is convex (concave) on some neighbourhood of 0 (1).

Let $\mathcal{A}$ denote the Borel-σ-field on $[0, 1]$ and let λ be Lebesgue measure on $\mathcal{A}$. As proved in [T1], [T2], T is conservative and exact w.r.t. λ and preserves a σ-finite measure μ equivalent to λ. The density $d\mu/d\lambda$ has a version h of the form

$$h(x) = h_0(x) \frac{x(1-x)}{(x - f_0(x))(f_1(x) - x)}, \quad x \in (0, 1),$$

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where $f_0 := (T\mid_{(0,c)})^{-1}$, $f_1 := (T\mid_{(c,1)})^{-1}$, and h_0 is continuous and positive on $[0,1]$. Maps of this type are known to have further strong ergodic properties, see e.g. [A0], [A2], [T3].

We will be interested in occupation times of neighbourhoods A, B of the indifferent fixed points $x = 0, 1$. As the invariant measure of $[0,1] \setminus (A \cup B)$ is finite, almost all orbits spend most of their time in $A \cup B$ (i.e. $n^{-1} \sum_{k=0}^{n-1} 1_{A\cup B} \circ T^k \to 1$ a.e.), and we investigate the asymptotic behaviour of $\sum_{k=0}^{n-1} 1_A \circ T^k$. When taken sufficiently small, the neighbourhoods A, B are dynamically separated in the sense of the following definition.

Dynamical separation. Let T be a map on X. Two disjoint sets $A, B \subset X$ are said to be *dynamically separated by* $Y \subset X$ if $x \in A$ (resp. B) and $T^n x \in B$ (resp. A) imply the existence of some $k = k(x) \in \{0, \ldots, n\}$ for which $T^k x \in Y$ (i.e. T-orbits can't pass from one set to the other without visiting Y).

Remark 1. *a) Let T be an interval map as above. Small neighbourhoods of the indifferent fixed points are dynamically separated by the interval $[f_0(c), f_1(c)]$.*

b) If the sets A, B are dynamically separated by Y, then so are any subsets $A' \subseteq A$, $B' \subseteq B$.

Distributional convergence. If ν is a probability measure on the measurable space $(X, \mathcal{A})$ and $(R_n)_{n\geq 1}$ is a sequence of measurable real functions on X, distributional convergence of $(R_n)_{n\geq 1}$ w.r.t. ν to some random variable R will be denoted by $R_n \overset{\nu}{\Longrightarrow} R$. *Strong distributional convergence* $R_n \overset{\mathcal{L}(\mu)}{\Longrightarrow} R$ on the σ-finite measure space $(X, \mathcal{A}, \mu)$ means that $R_n \overset{\nu}{\Longrightarrow} R$ for all probability measures $\nu \ll \mu$. If T is a nonsingular ergodic transformation on $(X, \mathcal{A}, \mu)$, compactness implies that if $R_n \circ T - R_n \overset{\mu}{\longrightarrow} 0$, then $R_n \overset{\mathcal{L}(\mu)}{\Longrightarrow} R$ as soon as $R_n \overset{\nu}{\Longrightarrow} R$ for some $\nu \ll \mu$ (compare section 3.6 of [A0] or [A1]).

We let $\mathcal{R}_\alpha$ denote the collection of functions regularly varying of index α at infinity, and interpret sequences $(a_n)_{n\geq 0}$ as functions on $\mathbb{R}_+$ via $t \longmapsto a_{[t]}$. For $\alpha \in (0,1)$, $\mathcal{G}_\alpha$ denotes a random variable distributed according to the *one-sided stable law of order* α, characterized by its Laplace transform $\mathbb{E}[\exp(-t\mathcal{G}_\alpha)] = e^{-t^\alpha}$, $t > 0$, and $\mathcal{G}_1 := 1$. Then the distribution of the variable $\mathcal{Y}_\alpha := \Gamma(1+\alpha)\,\mathcal{G}_\alpha^{-\alpha}$, $\alpha \in (0,1]$, is the *normalized Mittag-Leffler law of order* α (see section 3.6 of [A0]).

If T is some c.e.m.p.t. of the σ-finite measure space $(X, \mathcal{A}, \mu)$ and $M \in \mathcal{A}$, $\mu(M) > 0$, the *return time* function of M under T, defined as $\varphi_M(x) := \min\{n \geq 1 : T^n x \in M\}$, $x \in M$, is finite a.e., and the *induced map* $T_M : M \to M$, $T_M x := T^{\varphi_M(x)} x$, is a c.e.m.p.t. on $(M, \mathcal{A} \cap M, \mu\mid_{\mathcal{A}\cap M})$. There is a natural duality between the occupation times $S_n := \mathbf{S}_n(M)$ and the successive return times $\varphi_{M,n} := \sum_{k=0}^{n-1} \varphi_M \circ T_M^k$, $n \geq 1$, in that

$$S_k > n \Longleftrightarrow \varphi_{M,n} < k \qquad \text{on } M. \tag{1}$$

Whence, if $\alpha \in (0,1]$, $(a_n) \in \mathcal{R}_\alpha$, and $(b_n) \in \mathcal{R}_{\alpha^{-1}}$ is the asymptotic inverse of (a_n), then for any probability measure ν on $(M, \mathcal{A} \cap M)$,

$$\frac{1}{a_n} S_n \overset{\nu}{\Longrightarrow} \mathcal{Y}_\alpha \quad \text{iff} \quad \frac{1}{b_n} \varphi_{M,n} \overset{\nu}{\Longrightarrow} \mathcal{G}_\alpha. \tag{2}$$

By the Darling-Kac theorem for measure preserving transformations (cf. [A0], [A1]), this is what happens if $T : [0,1] \to [0,1]$ satisfies (1)-(3) with $Tx = x +$

$x^{1+p_0}\ell_0(x)$ and $1-T(1-x) = x + x^{1+p_1}\ell_1(x)$ near 0^+ with $p_0, p_1 \geq 1$ and ℓ_0, ℓ_1 slowly varying, and $\alpha := \max(p_0, p_1)^{-1}$, provided that $\mu(M) < \infty$. We show that this behaviour may persist for certain infinite measure sets M:

Theorem 1 (Distributional limits for barely infinite cusps)**.** *Let $T : [0,1] \to [0,1]$ satisfy (1)-(3), and assume that $Tx = x + x^{1+p_0}\ell_0(x)$ and $1-T(1-x) = x + x^2\ell_1(x)$ near 0^+ with $p_0 \geq 1$ and ℓ_0, ℓ_1 slowly varying, Then*

$$\frac{1}{c(n)} \sum_{k=0}^{n-1} 1_M \circ T^k \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{Y}_\alpha$$

for any $M \in \mathcal{A}$ with $\mu(M \bigtriangleup (c,1)) < \infty$, where $\alpha := p_0^{-1}$, and $c \in \mathcal{R}_\alpha$ is defined as $c(t) := \widetilde{a}^{-1}\left(\frac{t}{\Gamma(2-\alpha)\Gamma(1+\alpha)}[\sum_{k=0}^{t-1}(\theta^+ f_0^k(1) + \theta^-(1-f_1^k(0)))]^{-1}\right)$ with $\theta^\pm := 1/(T'(c^\pm))$, and $\widetilde{a}^{-1}$ asymptotically inverse to $\widetilde{a}(t) := t/[\theta^- \sum_{k=0}^{t-1}(1-f_1^k(0))]$, $t \geq 1$.

Weak law of large numbers for cusp visits. Notice that in case $p_0 = \alpha = 1$ we have $\mathcal{Y}_\alpha = 1$ and the theorem therefore provides us with a weak law of large numbers for this situation. In the balanced case (i.e. if $1 - T(1-x) \sim a^{-2}(Tx - x)$ as $x \to 0^+$ for some $a \in (0,\infty)$), this weak law is contained in [T5].

Example 1 (The standard examples of indifferent fixed points)**.** *If $Tx = x + a_0x^{1+p_0} + o(x^{1+p_0})$ and $1-T(1-x) = x + a_1x^2 + o(x^2)$ near 0^+ with $p_0 \geq 1$, then (again writing $\alpha := p_0^{-1}$) we find that*

$$c(n) \sim \begin{cases} \frac{\theta^-}{a_1}\left(\frac{\theta^+}{a_0} + \frac{\theta^-}{a_1}\right)^{-1} \cdot n & \text{if } p_0 = 1 \\ \frac{\alpha^{1-\alpha}(1-\alpha)}{\Gamma(2-\alpha)\Gamma(1+\alpha)} \frac{\theta^-}{\theta^+} \frac{a_0^\alpha}{a_1} \cdot n^\alpha \log n & \text{if } p_0 > 1. \end{cases}$$

To see this, recall (cf. [T2]*) that as $n \to \infty$, $\sum_{k=0}^{n-1}(1 - f_1^k(0)) \sim a_1^{-1} \cdot \log n$, and*

$$\sum_{k=0}^{n-1} f_0^k(1) \sim \begin{cases} a_0^{-1} \cdot \log n & \text{if } p_0 = 1 \\ \frac{1}{1-\alpha}\left(\frac{\alpha}{a_0}\right)^\alpha \cdot n^{1-\alpha} & \text{if } p_0 > 1. \end{cases}$$

Our second result concerns the pointwise behaviour of the ratios $\mathbf{S}_n(A)/\mathbf{S}_n(B)$ where A, B are neighbourhoods of the two fixed points. It shows (e.g.) that the weak lag of large numbers for cusp visits has no strong version (unless both cusps have finite measure) and extends some earlier results in this direction (compare [In1], [In2] and [AN]).

Theorem 2 (Almost sure divergence of occupation time ratios)**.** *Let $T : [0,1] \to [0,1]$ satisfy (1)-(3), and consider $A := [0, \delta_A)$, $B := (1-\delta_B, 1]$, $\delta_A, \delta_B \in (0,1)$.*

a) *In any case,*

$$\varliminf_{n\to\infty} \frac{\mathbf{S}_n(A)}{\mathbf{S}_n(B)} = 0 \text{ a.e.} \quad \text{or} \quad \varlimsup_{n\to\infty} \frac{\mathbf{S}_n(A)}{\mathbf{S}_n(B)} = \infty \text{ a.e.} \quad \text{(or both).}$$

b) *If $Tx - x = O(1 - x - T(1-x))$ as $x \to 0^+$, then*

$$\varlimsup_{n\to\infty} \frac{\mathbf{S}_n(A)}{\mathbf{S}_n(B)} = \infty \text{ a.e.}$$

In particular, if $Tx - x \asymp 1 - x - T(1-x)$ as $x \to 0^+$, then

$$\varliminf_{n\to\infty} \frac{\mathbf{S}_n(A)}{\mathbf{S}_n(B)} = 0 \text{ a.e.} \quad \text{and} \quad \varlimsup_{n\to\infty} \frac{\mathbf{S}_n(A)}{\mathbf{S}_n(B)} = \infty \text{ a.e.}$$

c) *If* $Tx = x + x^{1+p_0}\ell_0(x)$ *and* $1 - T(1-x) = x + x^{1+p_1}\ell_1(x)$ *near* 0^+ *with* $p_1 > p_0 > 1$ *and* ℓ_0, ℓ_1 *slowly varying, then*

$$\lim_{n\to\infty} \frac{\mathbf{S}_n(A)}{\mathbf{S}_n(B)} = 0 \ a.e.$$

In fact, the abstract result of section 4 below covers a few more subtle situations, we refer to the examples given there.

Observable measures. For $x \in [0,1]$ let $V_T(x)$ denote the set of accumulation points (in the space of Borel probability measures on $[0,1]$ equipped with weak convergence) of the empirical measures $\nu_n(x) := n^{-1}\sum_{k=0}^{n-1}\delta_{T^k x}$, $n \geq 1$. A Borel probability ν on $[0,1]$ is called *observable (for T)* if $\lambda(\{x : \nu \in V_T(x)\}) > 0$. It is an *SRB (Sinai-Ruelle-Bowen) measure (for T)* if $\lambda(\{x : V_T(x) = \{\nu\}\}) > 0$. By the ergodic theorem, since $\mu((\varepsilon, 1-\varepsilon)) < \infty$, we have $\nu_n((\varepsilon, 1-\varepsilon)) \to 0$ for any $\varepsilon \in (0, 1/2)$. Therefore, $\overline{\lim}_{n\to\infty}\mathbf{S}_n(A)/\mathbf{S}_n(B) = \infty$ a.e. implies $\delta_0 \in V_T(x)$ for a.e. $x \in [0,1]$, so that δ_0 is observable. If in addition $\underline{\lim}_{n\to\infty}\mathbf{S}_n(A)/\mathbf{S}_n(B) = 0$ a.e., then δ_1 is observable, too, and we have $V_T(x) = \{\rho\delta_0 + (1-\rho)\delta_1 : \rho \in [0,1]\}$ for a.e. $x \in [0,1]$. (As shown in [Z2], there are maps T satisfying (1)-(3) which exhibit similar behaviour even for $\nu_n := n^{-1}\sum_{k=0}^{n-1}\widetilde{\nu} \circ T^k$, $n \geq 1$, whenever $\widetilde{\nu}$ is a Borel probability absolutely continuous w.r.t. λ.) Finally, if $\lim_{n\to\infty}\mathbf{S}_n(A)/\mathbf{S}_n(B) = 0$ a.e., then δ_0 is the unique SRB measure for T.

2. A distributional limit theorem for barely infinite components

Let T be a c.e.m.p.t. of the σ-finite measure space $(X, \mathcal{A}, \mu)$. For the occupation times of sets $B \in \mathcal{A}$ under the action of m.p.t.s with sufficiently good mixing properties, distributional limit theorems have been obtained in the case that $\mu(B) < \infty$, cf. [A0], [A1], and in the case that $\mu(A) = \mu(B) = \infty$, where A, B are dynamically separated by a suitable set Y and there is very good balance between the return distributions to either side, cf. [T5]. Below we are going to discuss the asymptotic distributional behaviour , without any assumption on balance, but supposing that the component B is "barely infinite", meaning that we are at the borderline to finite measures. We show that distributionally the occupation times of such a set still behave as in the finite measure case as they converge (with different normalization though) to Mittag-Leffler laws. This generalizes the Darling-Kac limit theorem to certain sets of infinite measure.

Let S be some m.p.t. of the probability space $(\Omega, \mathcal{B}, P)$. A partition γ of Ω (mod P) will be called *continued-fraction (c.f.)-mixing* for S if it is nontrivial mod P and if $\infty > \psi_\gamma(n) \to 0$ as $n \to \infty$, where the *ψ-mixing coefficients* $\psi_\gamma(n)$, $n \geq 1$, of γ, are defined as

$$\psi_\gamma(n) := \sup\left\{\left|\log\frac{P(V\cap W)}{P(V)P(W)}\right| : k \geq 0, \begin{array}{ll} V \in \sigma(\bigvee_{j=0}^{k-1} S^{-j}\gamma), & P(V) > 0, \\ W \in S^{-n}\sigma(\bigvee_{j\geq 0} S^{-j}\gamma), & P(W) > 0 \end{array}\right\}.$$

Theorem 1 for interval maps is a special case of the following abstract distributional limit theorem for occupation times of barely infinite components dynamically separated from the rest of the space by some cyclic set with c.f.-mixing returns.

Theorem 3 (Distributional limits for barely infinite components). *Let T be a c.e.m.p.t. of the σ-finite measure space $(X, \mathcal{A}, \mu)$. Suppose that $X = A \cup B$ (disjoint), $\mu(A) = \mu(B) = \infty$, and $\mu(Y) < \infty$ with $Y := Y_A \cup Y_B := (B \cap T^{-1}A) \cup (A \cap T^{-1}B)$. Then Y dynamically separates A and B, and T_Y cyclically interchanges Y_A and Y_B.*

Assume that Y_A, Y_B, and the return time φ_Y are measurable w.r.t. some partition γ such that $\gamma_2 := \gamma \vee T_Y^{-1}\gamma$ is c.f.-mixing for $T_Y^2 \mid_{Y_A}$ and $T_Y^2 \mid_{Y_B}$. Let $L_A(t) := \int_{Y_A} (\varphi_Y \wedge t)\, d\mu$, and $L_B(t) := \int_{Y_B} (\varphi_Y \wedge t)\, d\mu$, $t > 0$. If $L_A \in \mathcal{R}_{1-\alpha}$, $\alpha \in (0, 1]$, and $L_B \in \mathcal{R}_0$, then for any $E \in \mathcal{A}$ with $\mu(E \bigtriangleup B) < \infty$,

$$\frac{1}{c(n)} \sum_{k=0}^{n-1} 1_E \circ T^k \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{Y}_\alpha,$$

where $c \in \mathcal{R}_\alpha$, $c(t) := a_B^{-1}\left(\frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)} \frac{t}{L_A(t)+L_B(t)}\right)$, $t \geq 1$, with a_B^{-1} asymptotically inverse to $a_B(t) := t/L_B(t)$.

Again the $\alpha = 1$ case provides us with weak laws of large numbers. Our result is flexible enough to cover situations in which weak laws with rather unusual normalization arise:

Example 2 (Weak law with oscillating normalizing sequences). *There are systems satisfying the assumptions of theorem 3 with $\alpha = 1$ for which*

$$\underline{\lim}_{n\to\infty} \frac{c(n)}{n} = 0 \quad \text{and} \quad \overline{\lim}_{n\to\infty} \frac{c(n)}{n} = 1. \tag{3}$$

To see this we construct suitable pairs of return distributions by specifying L_A and L_B. For any continuous increasing concave function $L > 0$ with $L(t)/t \to 0$ as $t \to \infty$, there is some $\mathbb{N}$-valued random variable φ for which $\mathrm{const} \cdot L(t) \sim \mathbb{E}[\varphi \wedge t]$ as $t \to \infty$. Assume that $L_A, L_B \in \mathcal{R}_0$ in addition satisfy $L_A(t), L_B(t) \nearrow \infty$, $\underline{\lim}_{t\to\infty} L_A(t)/L_B(t) = 0$, and $\overline{\lim}_{t\to\infty} L_A(t)/L_B(t) = \infty$. Define $a_B(t) := t/L_B(t)$ and $c(t) := a_B^{-1}(t/(L_A(t) + L_B(t)))$, then the uniform convergence theorem for regularly varying functions (cf. [BGT]*) implies (3). Therefore it is enough to construct L_A, L_B with the above properties.*

We are going to take $L_A(t) := \exp[\int_1^t \frac{\varepsilon_A(y)}{y}\, dy]$, $t \geq 1$, with a suitable decreasing piecewise constant function $\varepsilon_A : [1, \infty) \to (0, 1)$, $\varepsilon_A(y) = \sum_{n\geq 1} K_A(n) \cdot 1_{[t_n, t_{n+1})}(y)$ with $K_A(n) \in (0, 1)$, $K_A(n) \searrow 0$, $1 = t_1 < t_2 < \ldots < t_n \nearrow \infty$, and analogously for L_B. Then L_A, L_B are continuous, strictly increasing, and slowly varying. The required oscillation property will imply that $L_A(t), L_B(t) \nearrow \infty$. It is easily seen that functions of this type are concave.

For example, we may take $K_A(2n) := K_A(2n + 1) := (2n + 2)^{-1}$ and $K_A(2n + 1) := K_A(2n + 2) := (2n + 3)^{-1}$ for $n \geq 0$, and inductively define the t_n as follows. If, for some $n \geq 0$, $t_1, \ldots, t_{2n+1}$ have been constructed, we choose $t_{2n+2} > t_{2n+1}$ so large that

$$L_A(t_{2n+1})\, K_A(2n + 1)^{t_{2n+2}-t_{2n+1}} \geq n \cdot L_B(t_{2n+1})\, K_B(2n + 1)^{t_{2n+2}-t_{2n+1}},$$

which is possible since $K_A(2n+1) > K_B(2n+1)$. Then $L_A(t_{2n+2}) \geq n \cdot L_B(t_{2n+2})$. Analogously, if for some $n \geq 1$, $t_1, \ldots, t_{2n}$ have been constructed, we choose $t_{2n+1} > t_{2n}$ so large that

$$L_A(t_{2n})\, K_A(2n)^{t_{2n+1}-t_{2n}} \leq n^{-1} \cdot L_B(t_{2n})\, K_B(2n)^{t_{2n+1}-t_{2n}},$$

and hence $L_A(t_{2n+1}) \leq n^{-1} \cdot L_B(t_{2n})$.

As a preparation for the proof of the theorem, we now recall a few important facts about wandering rates.

Remark 2 (**Basic properties of wandering rates**). *Let T be a c.e.m.p.t. of the σ-finite measure space $(X, \mathcal{A}, \mu)$, $\mu(X) = \infty$. Recall (see e.g. section 3.8 of* [A0]*) that the* wandering rate *of a set $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, under T is the sequence defined by $w_n(Y) := \mu(\bigcup_{k=0}^{n-1} T^{-k}Y)$, $n \geq 1$, which always satisfies $w_n(Y) \nearrow \infty$, $w_n(Y)/n \searrow 0$, and $w_{n+1}(Y) \sim w_n(Y)$ as $n \to \infty$. Its importance for probabilistic questions is obvious from the observation that it equals the truncated expectation of the return time φ_Y of Y: $w_n(Y) = \int_Y (\varphi_Y \wedge n)\, d\mu$, $n \geq 1$. The wandering rate depends on Y, and, given T, there are no sets with maximal rate. Still, T may have sets Y with* minimal wandering rate, *meaning that $\underline{\lim}_{n\to\infty} w_n(Z)/w_n(Y) \geq 1$ for all $Z \in \mathcal{A}$, $0 < \mu(Z) < \infty$. If this is the case, we let $\mathcal{W}(T) \subseteq \mathcal{A}$ denote the collection of sets which have minimal wandering rate under T, and simply write $(w_n(T))_{n\geq 1}$ for any representing sequence. Below we shall use the easy observation (implicitly used e.g. in* [A1] *and* [T2]*) that*

$$E, F \in \mathcal{W}(T) \Longrightarrow E \cup F \in \mathcal{W}(T). \tag{4}$$

To verify this, notice that $w_n(E \cup F) = w_n(E) + \mu(\bigcup_{k=0}^{n-1} T^{-k}F \setminus \bigcup_{k=0}^{n-1} T^{-k}E)$, $n \geq 1$. Since $w_n(E) \sim w_n(F)$, it is enough to check that the rightmost term is $o(w_n(F))$ as $n \to \infty$. Choose some $K \geq 0$ for which $\widetilde{F} := F \cap T^{-K}E$ has positive measure. Then $w_{n-K}(\widetilde{F}) \sim w_n(\widetilde{F}) \sim w_n(F)$ as $F \in \mathcal{W}(T)$. Now $\mu(\bigcup_{k=0}^{n-1} T^{-k}F \setminus \bigcup_{k=0}^{n-1} T^{-k}E) \leq \mu(\bigcup_{k=0}^{n-1} T^{-k}F \setminus \bigcup_{k=0}^{n-K-1} T^{-k}\widetilde{F}) = w_n(F) - w_{n-K}(\widetilde{F}) = o(w_n(F))$.

Proof of theorem 3. Assume w.l.o.g. that $\mu(Y_A) = 1$. Let us first consider the specific set $E := B \cup Y_B$. We are going to prove the equivalent dual statement

$$\frac{1}{d(n)} \sum_{k=0}^{n-1} \varphi_E \circ T_E^k \overset{\mu_{Y_A}}{\Longrightarrow} \mathcal{G}_\alpha, \tag{5}$$

where $d(n) := b(n/L_B(n))$, $n \geq 1$, with $b \in \mathcal{R}_{\frac{1}{\alpha}}$ asymptotically inverse to $n \mapsto (\Gamma(2-\alpha)\Gamma(1+\alpha))^{-1} \cdot n/(L_A(n) + L_B(n))$. (Throughout, φ_M always denotes the return time function of some set M under the original map T.) Let $N_n := \sum_{k=0}^{n-1} 1_{Y_A} \circ T_E^k$, $n \geq 1$, then

$$\sum_{j=0}^{N_n-2} \varphi_{Y_A} \circ T_{Y_A}^j \leq \sum_{k=0}^{n-1} \varphi_E \circ T_E^k \leq \sum_{j=0}^{N_n-1} \varphi_{Y_A} \circ T_{Y_A}^j \quad \text{on } Y_A,$$

since $\sum_{k=\tau_j}^{\tau_{j+1}-1} \varphi_E \circ T_E^k = \varphi_{Y_A} \circ T_{Y_A}^j$ on Y_A for $j \geq 0$, where τ is the return time of Y_A under the action of T_E, $\tau_0 := 0$, and $\tau_j := \sum_{i=0}^{j-1} \tau \circ T_{Y_A}^i$, $j \geq 1$. Therefore, (5) follows at once if we show that for $i \in \{1, 2\}$,

$$\frac{1}{d(n)} \sum_{j=0}^{N_n-i} \varphi_{Y_A} \circ T_{Y_A}^j \overset{\mu_{Y_A}}{\Longrightarrow} \mathcal{G}_\alpha. \tag{6}$$

We verify (6) using

$$\frac{1}{b(n)} \sum_{j=0}^{n-i} \varphi_{Y_A} \circ T_{Y_A}^j \overset{\mu_{Y_A}}{\Longrightarrow} \mathcal{G}_\alpha \tag{7}$$

for $i \in \{1,2\}$, and

$$\frac{L_B(n)}{n}\,N_n \overset{\mu_{Y_A}}{\Longrightarrow} 1. \tag{8}$$

For the moment, assume (7) and (8), which will be proved below. Fix $\varepsilon > 0$ and take any $t > 0$, $t \notin \{1\} \cup \{1 - m^{-1} : m \geq 1\}$. (Then t is a point of continuity for the distribution function of each $(1-m^{-1})^{\frac{1}{\alpha}}\mathcal{G}_\alpha$, $m \geq 1$, $\alpha \in (0,1]$, and of $\mathcal{G}_1$.) Choose an integer so large that $\Pr[(1-m^{-1})^{\frac{1}{\alpha}}\mathcal{G}_\alpha \leq t] \leq \Pr[\mathcal{G}_\alpha \leq t] + \varepsilon$, and $n_0 = n_0(\varepsilon, m)$ so large that for $n \geq n_0$,

$$\mu_{Y_A}\left(\left\{1 - \frac{L_B(n)}{n}\,N_n > \frac{1}{m}\right\}\right) \leq \varepsilon,$$

as well as

$$\mu_{Y_A}\left(\left\{\frac{1}{b(n)}\sum_{j=0}^{(1-m^{-1})n-i} \varphi_{Y_A} \circ T_{Y_A}^j \leq t\right\}\right) \leq \Pr\left[\left(1 - \frac{1}{m}\right)^{\frac{1}{\alpha}} \mathcal{G}_\alpha \leq t\right] + \varepsilon.$$

For $n \geq n_0$ so large that also $n/L_B(n) \geq n_0$, we find

$$\begin{aligned}
&\mu_{Y_A}\left(\left\{\frac{1}{b(n/L_B(n))}\sum_{j=0}^{N_n-i} \varphi_{Y_A} \circ T_{Y_A}^j \leq t\right\}\right) \\
&\leq \mu_{Y_A}\left(\left\{1 - \frac{L_B(n)}{n}\,N_n > \frac{1}{m}\right\}\right) \\
&+ \mu_{Y_A}\left(\left\{\frac{1}{b(n/L_B(n))}\sum_{j=0}^{(1-m^{-1})n/L_B(n)-i} \varphi_{Y_A} \circ T_{Y_A}^j \leq t\right\}\right) \\
&\leq 2\varepsilon + \Pr\left[\left(1 - \frac{1}{m}\right)^{\frac{1}{\alpha}} \mathcal{G}_\alpha \leq t\right] \leq 3\varepsilon + \Pr[\mathcal{G}_\alpha \leq t].
\end{aligned}$$

The corresponding lower estimate is proved analogously, and we obtain (6).

It remains to check (7) and (8). The return time φ_{Y_A} is measurable $Y_A \cap \gamma_2$, which is a c.f.-mixing partition for $T_{Y_A} = T_Y^2\mid_{Y_A}$. Therefore the return-time process $(\varphi_{Y_A} \circ T_{Y_A})_{n\geq 0}$ of Y_A under T is c.f.-mixing. Hence, by lemma 3.7.4 of [A0], Y_A is a Darling-Kac set for T (and so is Y_B). According to the Darling-Kac limit theorem (cf. corollary 3.7.3 of [A0]) and the asymptotic renewal equation (proposition 3.8.7 of [A0]), for any $f \in L_1^+(\mu)$,

$$\Gamma(2-\alpha)\Gamma(1+\alpha)\frac{w_n(Y_A)}{n}\sum_{k=0}^{n-1} f \circ T^k \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mu(f)\,\mathcal{Y}_\alpha, \tag{9}$$

provided that the wandering rate $(w_n(Y_A))_{n\geq 1}$ of Y_A is regularly varying of index $1-\alpha$, $\alpha \in [0,1]$. Being Darling-Kac sets for T, both Y_A and Y_B have minimal wandering rates, see theorem 3.8.3 of [A0], and hence $w_n(Y_A) \sim w_n(Y_B) \sim w_n(Y)$ as $n \to \infty$, cf. remark 2. Consequently, $w_n(Y_A) \sim w_n(Y) = \mu(\bigcup_{k=0}^{n-1} T^{-k}Y) \sim \int_Y (\varphi_Y \wedge n)\,d\mu = L_A(n) + L_B(n)$, and $L_A + L_B \in \mathcal{R}_{1-\alpha}$ by the assumptions of our theorem. Therefore (7), which is the dual version of (9) with $f := 1_{Y_A}$, is established. (By regular variation of b we may take any fixed $i \geq 1$ in (7).)

A similar argument proves (8): The induced map T_E is a c.e.m.p.t. on $(E, \mathcal{A} \cap E, \mu\,|_{\mathcal{A}\cap E})$; conservativity and ergodicity are the content of propositions 1.5.1 and 1.5.2 of [A0], for the invariance of $\mu\,|_{\mathcal{A}\cap E}$ in the general (i.e. possibly infinite) case, see e.g. [He]. The return time τ of Y_A under T_E is measurable γ_2 and $(T_E)_{Y_A} = T_{Y_A}$. Therefore, the return process of Y_A under T_E is c.f.-mixing which (as before) implies that Y_A ia a Darling-Kac set for T_E. Since $\tau = 1 + \varphi_Y \circ T_Y$ on Y_A, and $\mu\,|_{\mathcal{A}\cap Y}$ is invariant under T_Y, the wandering rate of Y_A under T_E is given by

$$\begin{aligned}
\mu\left(\bigcup_{k=0}^{n-1} T_E^{-k} Y_A\right) &= \sum_{k=0}^{n-1} \mu\left(Y_A \cap \{\tau > k\}\right) \\
&= \mu(Y_A) + \sum_{k=0}^{n-2} \mu\left(Y_A \cap T_Y^{-1}\{\varphi_Y > k\}\right) \\
&= \mu(Y_A) + \sum_{k=0}^{n-2} \mu\left(Y_B \cap \{\varphi_Y > k\}\right) \\
&= \mu(Y_A) + L_B(n-1) \sim L_B(n).
\end{aligned}$$

Again using proposition 3.8.7 and corollary 3.7.3 of [A0] we obtain (8).

To finally pass to arbitrary sets $F \in \mathcal{A}$ with $\mu(F \bigtriangleup B) < \infty$ (equivalently $\mu(F \bigtriangleup E) < \infty$), take $f := 1_{E \setminus F}$ and $f := 1_{F\setminus E}$ in (9). Since $a_B(t) = o(t)$ implies $t = o(a_B^{-1}(t))$ as $t \to \infty$, the normalizing sequence in (9) is $o(c(n))$ as $n \to \infty$. We therefore see that $c(n)^{-1} \sum_{k=0}^{n-1} (1_E - 1_F) \circ T^k \xrightarrow{\mu} 0$ as $n \to \infty$, completing the proof of the theorem. $\square$

3. Sums versus maxima for nonintegrable c.f.-mixing processes

Our proof of almost sure divergence of the ratios in theorem 2 and its more general abstract version 5 below depends on the following result which is of considerable interest in itself.

Theorem 4 (Sums vs maxima for nonintegrable c.f.-mixing processes)**.** *Let γ be a c.f.-mixing partition for the m.p.t. S of the probability space $(\Omega, \mathcal{B}, P)$. Suppose that $\varphi, \psi : \Omega \to [0, \infty)$ are measurable γ with $\int_\Omega \varphi\, dP = \infty$. Let $L_\psi(t) := \int_\Omega (\psi \wedge t)\, dP$, $a_\psi(t) := t/L_\psi(t)$, $t > 0$.*

If $\int_\Omega a_\psi \circ \varphi\, dP = \infty$ (e.g. if $L_\psi(t) = O(L_\varphi(t))$ as $t \to \infty$), then

$$\overline{\lim_{n\to\infty}} \frac{\varphi \circ S^n}{\sum_{k=0}^{n-1} \psi \circ S^k} = \infty \qquad \text{a.e. on } \Omega. \tag{10}$$

Otherwise, i.e. if $\int_\Omega a_\psi \circ \varphi\, dP < \infty$, we have

$$\lim_{n\to\infty} \frac{\varphi \circ S^n}{\sum_{k=0}^{n-1} \psi \circ S^k} = 0 \qquad \text{a.e. on } \Omega. \tag{11}$$

The corresponding result for the case of iid sequences and $\varphi = \psi$ can be found in [Ke]. Let us look at a few specific examples for the theorem.

Example 3. *Observe that in the theorem φ may have a strictly lighter tail than ψ: Suppose for example that $P[\psi = n] \sim \kappa_\psi \cdot n^{-2}$ while $P[\varphi = n] \sim \kappa_\varphi \cdot (n^2 \log\log n)^{-1}$ as $n \to \infty$, then $L_\varphi(t) = o(L_\psi(t))$ as $t \to \infty$, but still $\int_\Omega \frac{\varphi}{L_\psi \circ \varphi}\, dP = \infty$, as Abel's*

series $\sum_{n\geq 1}(n \log n \log\log n)^{-1}$ *diverges. Analogous examples with heavier tails are obtained by taking* $P[\psi = n] \sim \kappa_\psi \cdot n^{-(1+\alpha)}$, $\alpha \in (0,1)$, *and* $P[\varphi = n] \sim \kappa_\varphi \cdot n^{-(1+\alpha)}(\log n)^{-1}$ *as* $n \to \infty$.

We are going to use Rényi's extension of the Borel-Cantelli lemma (cf. [Re]).

Lemma 1 (Rényi's Borel-Cantelli Lemma)**.** *Assume that* $(E_n)_{n\geq 1}$ *is a sequence of events in the probability space* $(\Omega, \mathcal{B}, P)$ *for which there is some* $r \in (0,\infty)$ *such that*

$$\frac{P(E_j \cap E_k)}{P(E_j)\,P(E_k)} \leq r \qquad \text{whenever } j,k \geq 1,\ j \neq k.$$

Then $P(\{E_n \text{ infinitely often}\}) > 0$ *iff* $\sum_{n\geq 1} P(E_n) = \infty$.

Proof of theorem 4. Notice first that by passing to $[\varphi]+1$ and $[\psi]+1$ we may assume w.l.o.g. that φ, ψ are integer-valued. We set $\psi_n := \sum_{k=0}^{n-1} \psi \circ S^k$, $n \geq 1$, and $a_\psi(t) := t/L_\psi(t)$, $t > 0$, and analogously for φ. Then $L_\psi(t), a_\psi(t) \nearrow \infty$ as $t \to \infty$, so that in particular $a_\psi(s+t) \leq a_\psi(s) + a_\psi(t)$ for $s,t > 0$, which shows that

$$\int_\Omega a_\psi \circ \varphi \; dP = \infty \quad \text{ iff } \quad \int_\Omega a_\psi \circ (c\varphi)\; dP = \infty \text{ for any } c > 0.$$

Moreover, $\int_\Omega a_\varphi \circ \varphi \, dP = \infty$ since $\int_\Omega \varphi \, dP = \infty$.

(i) We begin by showing that the stochastic order of magnitude of ψ_k is essentially given by $b_\psi(k)$, where b_ψ denotes the inverse function of a_ψ, defined on some (s_0, ∞), and satisfying $b_\psi(s) = sL_\psi(b_\psi(s))$. We claim that for t sufficiently small, there is some $\eta(t) > 0$ such that

$$P\left(\left\{\psi_k \leq b_\psi\left(\frac{k}{t}\right)\right\}\right) \geq \eta(t) \qquad \text{for all } k \geq 1. \tag{12}$$

To see this, let $(X, \mathcal{A}, \mu, T)$ be the conservative ergodic infinite measure preserving tower above $(\Omega, \mathcal{B}, P, S)$ with height function ψ, so that $\mu\mid_{\mathcal{A}\cap\Omega} = P$, $S = T_\Omega$, and ψ is the return time of Ω under T. By assumption, the return process $(\psi \circ S^n)_{n\geq 0}$ of Ω is c.f.-mixing, so that (by lemma 3.7.4 of [A0]) Ω is a Darling-Kac set for T. For $n \geq 1$ we let

$$N_n := \sum_{k=0}^{n-1} 1_\Omega \circ T^k \quad \text{ and } \quad a_n := \int_\Omega N_n \, d\mu \ .$$

The proof of proposition 3.7.1 of [A0] shows that $K := \sup_{n\geq 1} \int_\Omega \left(a_n^{-1} N_n\right)^2 d\mu < \infty$. Moreover, lemma 3.8.5 there implies that $r := \sup_{n\geq 1} a_\psi(n)/a_n < \infty$. For $t \in (0,1)$ and any $n \geq 1$ we therefore have

$$1 - t \leq \int_\Omega 1_{\{N_n \geq t a_n\}} \frac{N_n}{a_n}\, d\mu \leq \sqrt{K} \cdot \sqrt{\mu\left(\Omega \cap \{N_n \geq t a_n\}\right)}\ ,$$

and hence, if $t < r^{-1}$,

$$\mu\left(\Omega \cap \{N_n \geq t a_\psi(n)\}\right) \geq \mu\left(\Omega \cap \{N_n \geq tr a_n\}\right) \geq \frac{(1-rt)^2}{K} =: \eta(t)\ .$$

However, $N_n \geq t a_\psi(n)$ iff $\psi_{t a_\psi(n)} \leq n$, which proves (12).

(ii) Now fix any $N \geq 1$. In order to prove $\overline{\lim\limits_{n\to\infty}} \frac{\varphi\circ S^n}{\psi_n} \geq N$ a.s., we take any $t \in (0, r^{-1})$ and define

$$A_n := \Omega \cap \left\{ \rho_n \frac{\varphi \circ S^n}{\psi_n} \geq N \right\},$$

and

$$B_n := \Omega \cap \{\varphi \circ S^n \geq N\, b_\psi(n/t)\}, \quad C_n := \Omega \cap \{\psi_n \leq \, b_\psi(n/t)\}.$$

For arbitrary $n \geq 1$ we then have

$$\overline{A}_n := B_n \cap C_n \subseteq A_n.$$

Let $R := \psi_\gamma(1)$, the first ψ-mixing coefficient of γ. By c.f.-mixing, $R < \infty$, and $e^{-R} \leq P(\overline{A}_n)/(P(B_n)P(C_n)) \leq e^R$. According to (12), we have $P(C_n) \geq \eta(t) =: \eta > 0$. We are going to show that

$$P\left(\left\{\sum_{n\geq 1} 1_{\overline{A}_n} = \infty\right\}\right) > 0\ , \tag{13}$$

which immediately implies $\overline{\lim\limits_{n\to\infty}} \frac{\varphi\circ S^n}{\psi_n} \geq N$ a.e. on Y (since this limit function is T_Y-invariant), thus completing the proof of the proposition. To do so, we use lemma 1. Notice first that if $j \neq k$, then

$$\begin{aligned}
P\left(\overline{A}_j \cap \overline{A}_k\right) &\leq P\left(B_j \cap B_k\right) \leq e^R P\left(B_j\right) P\left(B_k\right) \\
&\leq e^{3R} \frac{P(\overline{A}_j)P(\overline{A}_k)}{P(C_j)P(C_k)} \\
&\leq \eta^{-2} e^{3R} P(\overline{A}_j)P(\overline{A}_k)\ ,
\end{aligned}$$

so that we are in fact in the situation of Rényi's Borel-Cantelli lemma, and it remains to check that $\sum_{n\geq 1} P(\overline{A}_n) = \infty$. By our previous observations and S-invariance of P,

$$\begin{aligned}
\sum_{n\geq 1} P(\overline{A}_n) &\geq \eta\, e^{-R} \sum_{n\geq 1} P(B_n) \\
&= \eta\, e^{-R} \sum_{n\geq 1} P\left(\left\{t\, a_\psi\left(\frac{\varphi}{N}\right) \geq n\right\}\right) \\
&\geq \eta\, e^{-R} t \cdot \left(\int_\Omega a_\psi\left(\frac{\varphi}{N}\right)\, dP - 1\right) = \infty,
\end{aligned}$$

proving (10). If $L_\psi(t) = O(L_\varphi(t))$ as $t \to \infty$, then $a_\varphi = O(a_\psi)$, and $\int_\Omega a_\varphi \circ \varphi\, dP = \infty$ whenever $\int_\Omega \varphi\, dP = \infty$.

(iii) To prove the converse, assume that $\int_\Omega a_\psi \circ \varphi\; dP < \infty$, then $\sum_{j\geq 1} P(\{\varphi = j\}) \cdot a_j < \infty$ as well (use lemma 3.8.5 of [A0] again). Observe also that $a_j = \sum_{n\geq 0} P(\{\psi_n < j\})$. Now

$$\begin{aligned}
P(\{\varphi \circ S^n > \psi_n\}) &= \sum_{j\geq 1} P(\{\varphi \circ S^n = j \text{ and } \psi_n < j\}) \\
&\leq e^R \sum_{j\geq 1} P(\{\varphi = j\}) \cdot P(\{\psi_n < j\}),
\end{aligned}$$

and hence

$$\sum_{n\geq 1} P(\{\varphi\circ S^n > \psi_n\}) \leq e^R \sum_{j\geq 1} P(\{\varphi = j\}) \cdot \sum_{n\geq 1} P(\{\psi_n < j\})$$
$$\leq e^R \sum_{j\geq 1} P(\{\varphi = j\}) \cdot a_j < \infty.$$

By Borel-Cantelli we therefore see that $\overline{\lim}_{n\to\infty} \varphi\circ S^n / \sum_{k=0}^{n-1} \psi\circ S^k \leq 1$ a.e., and since the same argument applies also applies to $c\varphi$ for any $c>0$, our claim follows. □

4. Almost sure divergence of the ratios

Again, let T be a c.e.m.p.t. of the σ-finite measure space $(X,\mathcal{A},\mu)$. The ratios $\mathbf{S}_n(A)/\mathbf{S}_n(B)$ of occupation times of disjoint sets of infinite measure may well converge almost surely. This obviously happens in cyclic situations, take for example the sets A,B of even and odd integers for the (null-recurrent) coin-tossing random walk. In the examples we are mainly interested in (interval maps with indifferent fixed points) this trivial case cannot occur since the sets A,B are dynamically separated by some set $Y\in\mathcal{A}$ with $0<\mu(Y)<\infty$. Still, this condition is not enough to enforce almost sure divergence of the ratios, as the following Markov-chain example illustrates.

Example 4 (A renewal chain for which pointwise ratio limits do exist). *Let $(f_k)_{k\geq 1}$ be a probability distribution such that $\sum k f_k < \infty$ but $\sum k^2 f_k = \infty$. Consider the renewal chain $(X_n)_{n\geq 0}$ associated to (f_k), i.e. the Markov chain with state space $S := \{0,1,\ldots\}$ and transition probabilities $p_{0,k-1} = f_k$ and $p_{k,k-1} = 1$ for $k\geq 1$. This irreducible chain has an invariant probability distribution μ given by $\mu_k = \mu_0 \sum_{j>k} f_j$, $k\geq 0$. According to our moment assumption, $\mathbb{E}_\mu[X_n] = \infty$, that is, (X_n) is a stationary (under μ) sequence of nonnegative random variables with infinite expectation. Nevertheless,*

$$\lim_{n\to\infty} \frac{X_n}{n} = 0 \qquad a.s., \tag{14}$$

compare [Ta]*, example a). Let us then construct a tower above (X_n), i.e. a new chain $(\widetilde{X}_n)$ with state space $\widetilde{S} := \{(k,j) : k\in S, 0\leq j\leq 2k+1\}$ and transition probabilities $p_{(0,0),(k-1,0)} = f_k$, $p_{(k,j-1),(k,j)} = 1$ if $1\leq j\leq 2k+1$, and $p_{(k,2k+1),(k-1,0)} = 1$, $k\geq 1$. This again is a renewal chain. The stationary measure $\widetilde{\mu}$, given by $\widetilde{\mu}_{(k,j)} := \mu_k$ is infinite, i.e. $(\widetilde{X}_n)$ is null-recurrent. Let $Y := \{(k,j)\in\widetilde{S} : j=0 \text{ or } j=k+1\}$, which has finite measure and dynamically separates the two components $A := \{(k,j) : 0<j\leq k\}$ and $B := \{(k,j) : j>k+1\}$ of its complement. We claim that*

$$\lim_{n\to\infty} \frac{\mathbf{S}_n(A)}{\mathbf{S}_n(B)} = \lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} 1_A(\widetilde{X}_k)}{\sum_{k=0}^{n-1} 1_B(\widetilde{X}_k)} = 1 \qquad a.s.$$

Assume w.l.o.g. that $\widetilde{X}_0 = (0,0)$, then $\mid \mathbf{S}_n(A) - \mathbf{S}_n(B) \mid \leq X_{N_n}$, where $N_n := \sum_{k=1}^{n-1} 1_S(\widetilde{X}_k)$, $n\geq 1$. By (14), however, we have $X_{N_n} = o(N_n)$ a.s., and since $N_n = O(\mathbf{S}_n(B))$ a.s. (in fact $o(\mathbf{S}_n(B))$), the claim follows.

The proof of a.s. convergence in this example uses the very strong dependence between the respective durations of excursions to A and B. Below we show that a.s.

convergence in fact can no longer happen if there is enough independence between the excursions.

Theorem 5 (Almost sure divergence of occupation time ratios)**.** *Let T be a c.e.m.p.t. of the σ-finite measure space $(X, \mathcal{A}, \mu)$. Suppose that $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, dynamically separates $A, B \in \mathcal{A}$ with $X = A \cup B \cup Y$ (disjoint) and $\mu(A) + \mu(B) = \infty$.*

a) *Assume that the return time φ_Y is measurable w.r.t. some c.f.-mixing partition γ for T_Y, then*

$$\underline{\lim}_{n\to\infty} \frac{\mathbf{S}_n(A)}{\mathbf{S}_n(B)} = 0 \text{ a.e.} \qquad \text{or} \qquad \overline{\lim}_{n\to\infty} \frac{\mathbf{S}_n(A)}{\mathbf{S}_n(B)} = \infty \text{ a.e.} \qquad \text{(or both).}$$

Now suppose that $X = A \cup B$ (disjoint), $\mu(A) = \mu(B) = \infty$, and $\mu(Y) < \infty$ with $Y := Y_A \cup Y_B := (B \cap T^{-1}A) \cup (A \cap T^{-1}B)$. Assume that Y_A, Y_B, and the return time φ_Y are measurable w.r.t. some partition γ such that $\gamma_2 := \gamma \vee T_Y^{-1}\gamma$ is c.f.-mixing for $T_Y^2 \mid_{Y_A}$ and $T_Y^2 \mid_{Y_B}$. Let $L_A(t) := \int_{Y_A} (\varphi_Y \wedge t)\, d\mu$, and $L_B(t) := \int_{Y_B} (\varphi_Y \wedge t)\, d\mu$, $t > 0$.

b) *If $L_B(t) = O(L_A(t))$, then*

$$\overline{\lim}_{n\to\infty} \frac{\mathbf{S}_n(A)}{\mathbf{S}_n(B)} = \infty \text{ a.e.}$$

The same conclusion still holds if $\int_{Y_A} \frac{\varphi_Y}{L_B \circ \varphi_Y}\, d\mu = \infty$ and $L_A(t) = O(L_B(t))$.

c) *If $L_A \in \mathcal{R}_{1-\alpha}$, and $L_B \in \mathcal{R}_{1-\beta}$, with $0 < \beta < \alpha < 1$, then*

$$\lim_{n\to\infty} \frac{\mathbf{S}_n(A)}{\mathbf{S}_n(B)} = 0 \text{ a.e.}$$

The same conclusion still holds if $0 < \beta = \alpha < 1$ and $\int_{Y_B} a_A^* \circ \varphi_Y\, d\mu < \infty$, where a_A^* is the inverse of $b_A^*(t) := b_A(t/\log\log t) \cdot \log\log t$, $t > 0$.

Example 5. *To obtain an example for statement c) of the theorem with $\alpha = \beta$, choose return distributions with $\mu_{Y_A}[\varphi_Y = n] \sim \kappa_A \cdot n^{-(1+\alpha)}$ and $\mu_{Y_B}[\varphi_Y = n] \sim \kappa_B \cdot n^{-(1+\alpha)}(\log n)^{-2}$.*

Proof of Theorem 5. Assume w.l.o.g. that $\mu(Y) = 1$. For part a) of the theorem, denote $\varphi := \varphi_Y$, the return time of Y, $Y_A := Y \cap T^{-1}A$, $Y_B := Y \cap T^{-1}B$, and define

$$S_n^A := \sum_{k=0}^{n-1} 1_{A\cup Y_A} \circ T^k,\ S_n^B := \sum_{k=0}^{n-1} 1_{B\cup Y_B} \circ T^k, \text{ and } R_n := \frac{S_n^A}{S_n^B},\ n \geq \dot{1}.$$

Now if $T_Y^n x \in Y_A$, then $T^j x \in A$ for $j \in \{1, \ldots, \varphi(T_Y^n x) - 1\}$, so that

$$S^A_{\varphi_{n+1}(x)}(x) = S^A_{\varphi_n(x)}(x) + \varphi(T_Y^n x) \quad \text{and} \quad S^B_{\varphi_{n+1}(x)}(x) = S^B_{\varphi_n(x)}(x).$$

Consequently,

$$R_{\varphi_{n+1}(x)}(x) - R_{\varphi_n(x)}(x) = \frac{\varphi(T_Y^n x)}{S^B_{\varphi_n(x)}(x)} \geq \frac{\varphi(T_Y^n x)}{\varphi_n(x)}\ .$$

Interchanging the roles of A and B, we obtain an analogous estimate with R replaced by R^{-1} if $T_Y^n x \in Y_B$. Therefore,

$$\overline{R}(x) := \overline{\lim}_{n\to\infty} \left(R_{\varphi_{n+1}(x)}(x) - R_{\varphi_n(x)}(x)\right) \geq \overline{\lim}_{n\to\infty, T_Y^n x \in Y_A} \frac{\varphi \circ T_Y^n}{\varphi_n}(x) \qquad \text{a.e. on } Y,$$

and

$$\underline{R}(x) := \varlimsup_{n\to\infty} \left(R^{-1}_{\varphi_{n+1}(x)}(x) - R^{-1}_{\varphi_n(x)}(x) \right) \geq \varlimsup_{n\to\infty, T_Y^n x \in Y_B} \frac{\varphi \circ T_Y^n}{\varphi_n}(x) \qquad \text{a.e. on } Y.$$

According to our assumption, theorem 4 applies to the induced map T_Y to ensure that

$$\varlimsup_{n\to\infty} \frac{\varphi \circ T_Y^n}{\varphi_n} = \infty \qquad \text{a.e. on } Y, \tag{15}$$

where again $\varphi_n := \sum_{k=0}^{n-1} \varphi \circ T_Y^k$, $n \geq 1$. Since $\varphi = 1$ on $Y \setminus (Y_A \cup Y_B)$, the same is true along the subsequences where $T_Y^n x \in Y_A \cup Y_B$. Hence at least one of $\overline{R}$ and $\underline{R}$ is infinite a.e. on Y, and hence on X due to the T-invariance of these limit functions. Therefore $\varlimsup_{n\to\infty} R_n = \infty$ or $\varlimsup_{n\to\infty} R_n^{-1} = \infty$, or both, implying assertion a).

For the part b), let $\varphi := 1_{Y_A} \cdot \varphi_Y$ (so that $L_\varphi = L_A$) and $\psi := 1_{Y_A}(\varphi_Y + \varphi_Y \circ T_Y)$. We have

$$\begin{aligned} L_\psi(t) = & \int_0^t \mu\left(Y_A \cap \{\varphi_Y + \varphi_Y \circ T_Y \geq s\} \cap \{\varphi_Y \leq \varphi_Y \circ T_Y\}\right) ds \\ & + \int_0^t \mu\left(Y_A \cap \{\varphi_Y + \varphi_Y \circ T_Y \geq s\} \cap \{\varphi_Y > \varphi_Y \circ T_Y\}\right) ds, \end{aligned}$$

where $\mu\left(Y_A \cap \{\varphi_Y + \varphi_Y \circ T_Y \geq s\} \cap \{\varphi_Y \leq \varphi_Y \circ T_Y\}\right) \leq \mu\left(Y_A \cap \{2\varphi_Y \circ T_Y \geq s\}\right) \leq \mu\left(Y_B \cap \{\varphi_Y \geq \frac{s}{2}\}\right)$ and similarly $\mu\left(Y_A \cap \{\varphi_Y + \varphi_Y \circ T_Y \geq s\} \cap \{\varphi_Y > \varphi_Y \circ T_Y\}\right) \leq \mu\left(Y_A \cap \{\varphi_Y \geq \frac{s}{2}\}\right)$. Therefore,

$$L_\psi(t) \leq 2\left(L_B\left(\frac{t}{2}\right) + L_A\left(\frac{t}{2}\right)\right).$$

If $L_B(t) = O(L_A(t))$, then the righthand side is $O(L_\varphi(t))$. Otherwise, if $L_A(t) = O(L_B(t))$ and $\int_{Y_A} \frac{\varphi_Y}{L_B \circ \varphi_Y}\, d\mu = \infty$, then $\int_{Y_A} \frac{\varphi_Y}{L_\psi \circ \varphi_Y}\, d\mu = \infty$. According to theorem 4 therefore

$$\varlimsup_{n\to\infty} \frac{\varphi \circ S^n}{\sum_{k=0}^{n-1} \psi \circ S^k} = \infty \qquad \text{a.e. on } Y_A.$$

On the other hand, if $x \in Y_A$, then for all $n \geq 1$, $\mathbf{S}_{\psi_n(x) + \varphi \circ T_Y^{2n}(x)}(A \setminus Y_B)(x) = \mathbf{S}_{\psi_n(x)}(A \setminus Y_B)(x) + \varphi \circ T_Y^{2n}(x)$ while $\mathbf{S}_{\psi_n(x) + \varphi \circ T_Y^{2n}(x)}(B \setminus Y_A)(x) = \mathbf{S}_{\psi_n(x)}(B \setminus Y_A)(x) \leq \psi_n(x)$. This implies $\varlimsup_{n\to\infty} \mathbf{S}_n(A \setminus Y_B)/\mathbf{S}_n(B \setminus Y_A) = \infty$ a.e. and hence the assertion of part b).

Proof of part c) of the theorem. If $L_A \in \mathcal{R}_{1-\alpha}$, $\alpha \in (0,1)$, then $\mu_{Y_A}(\{\varphi_Y \geq t\}) \sim (1-\alpha)\, a_A(t)^{-1}$ as $t \to \infty$, and $a_A \in \mathcal{R}_\alpha$. Let $b_A \in \mathcal{R}_{\alpha^{-1}}$ be the inverse of a_A. According to theorem 5 of [AD],

$$\varliminf_{n\to\infty} \frac{1}{b_A^*(n)} \sum_{k=0}^{n-1} \varphi_Y \circ T_Y^{2k} = K(\alpha) \in (0, \infty) \text{ a.e. on } Y_A, \tag{16}$$

where $b_A^*(t) := b_A(t/\log\log t) \cdot \log\log t$ (and hence $b_A^* \in \mathcal{R}_{\alpha^{-1}}$). On the other hand, theorem 2.4.1 of [A0] implies that

$$\varlimsup_{n\to\infty} \frac{1}{b_A^*(n)} \sum_{k=0}^{n} \varphi_Y \circ T_Y^{2k+1} = \infty \qquad \text{a.e. on } Y_A \tag{17}$$

provided that $\int_{Y_A} a_A^* \circ \varphi_Y \circ T_Y \, d\mu = \int_{Y_B} a_A^* \circ \varphi_Y \, d\mu < \infty$, where a_A^* is the inverse of b_A^*. It is clear that (16) and (17) together give the desired result. If $\beta < \alpha$, then $\int_{Y_B} \varphi_Y^\rho \, d\mu < \infty$ for $\rho < \beta$, and hence $\int_{Y_B} a_A^* \circ \varphi_Y \, d\mu < \infty$. □

5. Application to interval maps with indifferent fixed points

We show how theorems 3 and 5 apply to the interval maps to yield theorems 1 and 2 advertized in the introduction.

Proof of theorem 1. We are going to apply theorem 3 with $A := (0, c)$ and $B := (c, 1)$. Standard arguments (compare [T1]) show that T_Y ia a uniformly expanding piecewise monotone map satisfying "Adler's condition", i.e. $T_Y''/(T_Y')^2$ is bounded. The return time function φ_Y is measurable w.r.t. the natural fundamental partition γ for T_Y. The image of any $W \in \gamma$ contained in Y_A equals Y_B and vice versa. Therefore γ_2 is c.f.-mixing for the restrictions of T_Y^2 to Y_A and Y_B.

As in the proof of theorem 3, $L_B(n) \sim \mu(\bigcup_{k=0}^{n-1} T_E^{-k} Y_A) \sim \mu(\bigcup_{k=0}^{n-1} T_E^{-k} Y)$, where $E := B \cup Y_B = (f_0(c), 1)$. However, it is easily seen that $\mu(\bigcup_{k=0}^{n-1} T_E^{-k} Y) = \mu(Y_B) + \mu(\bigcup_{k=0}^{n-1} T_B^{-k} Y_A) \sim w_n(T_B)$, and T_B is a map from the class studied in [T2]. Analogously, $L_A(n) \sim w_n(T_A)$. Lemma 5 of [T2] therefore shows that

$$L_A(n) \sim \frac{h_0(0)}{c} \sum_{k=0}^{n-1} f_0^k(1) \quad \text{and} \quad L_B(n) \sim \frac{h_0(1)}{c} \sum_{k=0}^{n-1} (1 - f_1^k(0)) \quad \text{as } n \to \infty.$$

According to lemma 3 (b) of [T4], $Tx = x + x^{1+p_0} \ell_0(x)$ near 0^+ thus implies $L_A \in \mathcal{R}_{1-\alpha}$. By the same argument, $L_B \in \mathcal{R}_0$, and theorem 3 applies. Notice further that $L_B(n) \sim \mu(\bigcup_{k=0}^{n-1} T_E^{-k} Y) \sim \sum_{k=0}^{n-1} \mu(Y_B \cap \{\varphi_Y > k\})$, and inspection of the map T and continuity of h show that $\mu(Y_B \cap \{\varphi_Y > k\}) \sim h(c) \, \lambda(Y_B \cap \{\varphi_Y > k\}) \sim (h(c)/T'(c^-)) \cdot (1 - f_1^k(0))$, and similarly for $L_A(n)$. Hence

$$\frac{h_0(0)}{c} = \frac{h(c)}{T'(c^+)} \quad \text{and} \quad \frac{h_0(1)}{c} = \frac{h(c)}{T'(c^-)},$$

which gives the constants for the normalizing sequence. □

The proof of theorem 2 uses the following observation.

Lemma 2 (Comparing different indifferent fixed points)**.** *Let $f, g : [0, \kappa] \to [0, \infty)$ be increasing $\mathcal{C}^1$-functions with $0 \le f(x), g(x) < x$ for $x \in (0, \kappa]$ and $f'(0) = g'(0) = 1$.*

a): *Assume that $x - f(x) = O(x - g(x))$ as $x \to 0^+$. Then*

$$\sum_{j=0}^{n-1} g^j(\kappa) = O\left(\sum_{j=0}^{n-1} f^j(\kappa) \right) \qquad \text{as } n \to \infty.$$

b): *Assume that $\sum_{j \ge 0} f^j(\kappa) = \infty$ with $x - f(x)$ regularly varying of index $1 + p$, $p \ge 1$, and that $x - f(x) \sim a^{-p}(x - g(x))$ for some $a \in (0, \infty)$ as*

$x \to 0^+$. *Then*

$$\sum_{j=0}^{n-1} g^j(\kappa) \sim \frac{1}{a} \sum_{j=0}^{n-1} f^j(\kappa) \qquad \textit{as } n \to \infty.$$

Proof. We only verify b), the proof of a) being an easier application of the same type of argument. Notice first that for any integer $q \geq 1$, $x - f^q(x) = \sum_{j=0}^{q-1}(f^j(x) - f^{j+1}(x)) = \sum_{j=0}^{q-1}(f^j)'(\xi_j) \cdot (x - f(x))$ for suitable $\xi_j \in (f(x), x)$, $j \in \{0, \ldots, q-1\}$. Since $(f^j)'(0) = 1$ for all $j \geq 0$, we therefore find that

$$\lim_{x \to 0^+} \frac{x - f^q(x)}{x - f(x)} = q \quad \text{for any } q \in \mathbb{N}. \tag{18}$$

Let us then observe that for any $m', m'', q \in \mathbb{N}$ and $x, y \in (0, \kappa]$,

$$\sum_{j=m'}^{n \pm m''} f^j(x) \sim \sum_{j=0}^{n-1} f^j(y) \quad \text{and} \quad \sum_{j=0}^{kq-1} f^j(x) \sim q \sum_{i=0}^{k-1} f^{iq}(x) \tag{19}$$

as $n \to \infty$. The first of these is trivial (using $\sum_{j \geq 0} f^j(x) = \infty$), for the second use the first and monotonicity of $(f^j(x))_{j \geq 0}$. Assume now that $C > 0$ and $\kappa_C \in (0, \kappa]$ are such that $x - f(x) \leq C(x - g(x))$ for $x \in (0, \kappa_C]$. Choose $q, r \in \mathbb{N}$ with $\frac{r}{q} > C$, and expand

$$\frac{x - f^q(x)}{x - g^r(x)} = \frac{x - f(x)}{x - g(x)} \frac{x - f^q(x)}{x - f(x)} \frac{x - g(x)}{x - g^r(x)},$$

to see by (18) that $x - f^q(x) < x - g^r(x)$ for x small, and hence

$$g^r(x) \leq f^q(x) \quad \text{for } x \in (0, \eta]. \tag{20}$$

Consequently, $g^{jr}(\eta) < f^{jq}(\eta)$ for all $j \geq 0$. According to (19) we therefore obtain

$$\sum_{j=0}^{n-1} g^j(\kappa) \sim r \sum_{i=0}^{[n/r]} g^{ir}(\eta) < r \sum_{i=0}^{[n/r]} f^{ir}(\eta) \sim \frac{r}{q} \sum_{j=0}^{q[n/r]} f^j(\eta) \sim \frac{r}{q} \sum_{j=0}^{qn/r} f^j(\kappa).$$

Notice that by lemma 3 of [T4], $(\sum_{k=0}^{n-1} f^k(\kappa))_{n \geq 1} \in \mathcal{R}_{1-\frac{1}{p}}$, so that $\sum_{j=0}^{qn/r} f^j(\kappa) \sim (r/q)^{\frac{1}{p}} \sum_{j=0}^{n-1} f^j(\kappa)$, and we get $\sum_{j=0}^{n-1} g^j(\kappa) \leq (1 + o(1)) \left(\frac{r}{q}\right)^{\frac{1}{p}} \sum_{j=0}^{n-1} f^j(\kappa)$. Since $C > a^{-p}$ and $r/q > C$ were arbitrary, we end up with

$$\sum_{j=0}^{n-1} g^j(\kappa) \leq (1 + o(1)) \cdot \frac{1}{a} \sum_{j=0}^{n-1} f^j(\kappa) \quad \text{as } n \to \infty,$$

and interchanging the roles of f and g completes the proof. $\square$

Proof of Theorem 2. For the first assertion we may w.l.o.g. take $A := [0, x_2)$, $B := (\widetilde{x}_2, 1]$, where x_2 is the unique point of period 2 in $(0, c)$, and $\widetilde{x}_2 := Tx_2$. Let $Y := [x_2, \widetilde{x}_2]$, then T_Y is a uniformly expanding piecewise onto map with countable fundamental partition γ, φ_Y is measurable γ, and standard arguments (compare [T1]) show that T_Y satisfies "Adler's condition", i.e. $T''/(T')^2$ is bounded. Therefore γ is c.f.-mixing for T_Y, and part a) of theorem 5 applies.

Turning to part b) and c), we choose $A := (0, c)$ and $B := (c, 1)$ as in the proof of theorem 1, where we found that this partition satisfies the assumptions of

parts b) and c) of theorem 5, and that $L_A(n) \sim const \sum_{k=0}^{n-1} f_0(1)$ and $L_B(n) \sim const \sum_{k=0}^{n-1}(1 - f_1(0))$. Assertion b) therefore follows from lemma 2. For part c) it is enough to recall that (as in the proof of 1) $Tx = x + x^{1+p}\ell(x)$ at 0^+ implies $L_A \in \mathcal{R}_{1-p^{-1}}$. Therefore theorem 5 c) applies. $\square$

Let us stress that the more subtle situations of nonequivalent rates L_A and L_B with the same index of regular variation as in examples 3 and 5 also occur in the present setup. Indeed, by arguments analogous to those of theorem 4.8.7 of [A0], given any $L_i \in \mathcal{R}_{\gamma_i}$, $\gamma_i \in (0, 1)$, $i \in \{0, 1\}$, there is some map T satisfying (1)-(3) for which $L_A(t)$ ($L_B(t)$) is asymptotically equivalent to $L_0(t)$ ($L_1(t)$) as $t \to \infty$.

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References

[A0] J.Aaronson: *An Introduction to Infinite Ergodic Theory.* AMS 1997.

[A1] J.Aaronson: *On the asymptotic distributional behaviour of transformations preserving infinite measures*. J. Anal. Math. **39** (1981), 203-234.

[A2] J.Aaronson: *Random f-expansions*. Ann. Probab. **14** (1986), 1037-1057.

[AD] J.Aaronson, M.Denker: *Upper bounds for ergodic sums of infinite measure preserving transformations*. Trans. Amer. Math. Soc. **319** (1990), 101-138.

[AN] J.Aaronson, H.Nakada: *Trimmed sums for non-negative mixing stationary processes.* Stochastic Processes and their Applications **104** (2003), 173-192.

[BGT] N.H.Bingham, C.M.Goldie, J.L.Teugels: *Regular Variation.* Cambridge University Press 1989.

[He] G.Helmberg: *Über konservative Transformationen*. Math. Annalen **165** (1966), 44-61.

[In1] T.Inoue: *Ratio ergodic theorems for maps with indifferent fixed points.* Ergod. Th. & Dynam. Sys. **17** (1997), 625-642. Erratum: Ergod. Th. & Dynam. Sys. **21** (2001), 1273.

[In2] T.Inoue: *Sojourn times in small neighbourhoods of indifferent fixed points of one-dimensional dynamical systems.* Ergod. Th. & Dynam. Sys. **20** (2000), 241-257.

[Ke] H.Kesten: *Solution to advanced problem 5716.* AMM **78** (1971), 305-308.

[Re] A.Rényi: *Probability theory.* North-Holland, Amsterdam 1970.

[Ta] D.Tanny: *A zero-one law for stationary sequences.* Z.Wahrscheinlichkeitstheorie verw. Geb. **30** (1974), 139-148.

[T1] M.Thaler: *Estimates of the invariant densities of endomorphisms with indifferent fixed points.* Israel J. Math. **37** (1980), 303-314.

[T2] M.Thaler: *Transformations on [0,1] with infinite invariant measures.* Israel J. Math. **46** (1983), 67-96.

[T3] M.Thaler: *A limit theorem for the Perron-Frobenius operator of transformations on [0,1] with indifferent fixed points.* Israel J. Math. **91** (1995), 111-127.

[T4] M.Thaler: *The Dynkin-Lamperti Arc-Sine Laws for Measure Preserving Transformations.* Trans. Amer. Math. Soc. **350** (1998), 4593-4607.

[T5] M.Thaler: *A limit theorem for sojourns near indifferent fixed points of one-dimensional maps*. Ergod. Th. & Dynam. Sys. **22** (2002), 1289-1312.

[Z1] R.Zweimüller: *Ergodic properties of infinite measure preserving interval maps with indifferent fixed points.* Ergod. Th. & Dynam. Sys. **20** (2000), 1519-1549.

[Z2] R.Zweimüller: *Exact $\mathcal{C}^\infty$ covering maps of the circle without (weak) limit measure*. Colloq. Math. **93** (2002), 295-302.

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